Attitude Representation and Transformation Matrices

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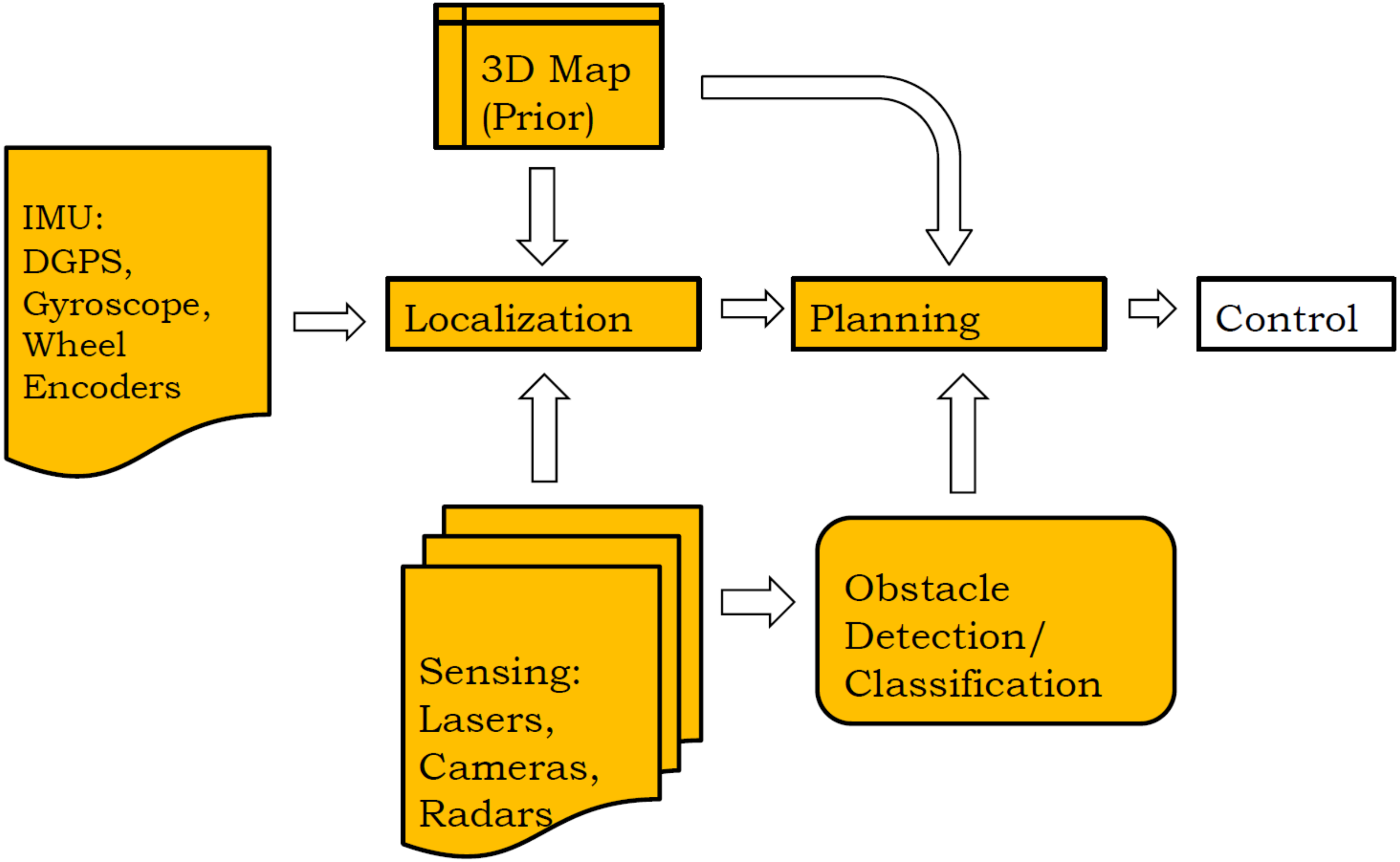
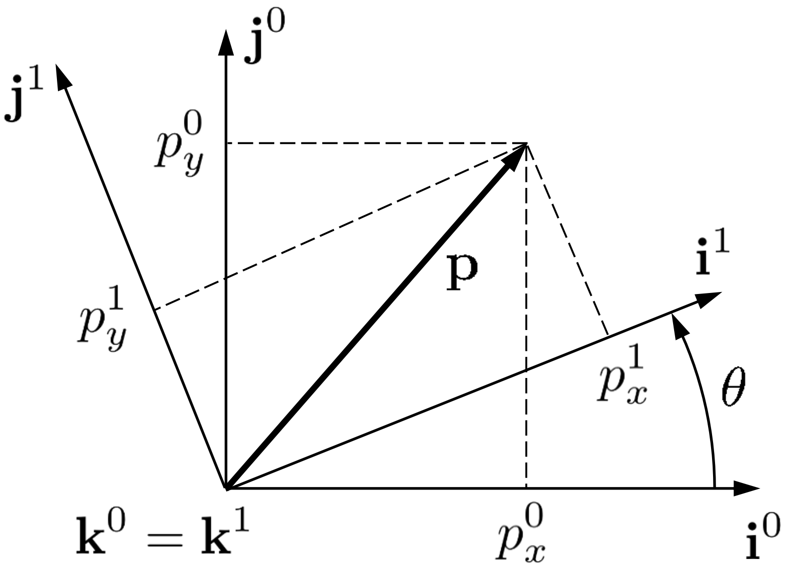
Autonomous Navigation System



Coordinated Frames

- Describe relative position and orientation of objects
  - Aircraft relative to direction of wind
  - Camera relative to aircraft
  - Aircraft relative to inertial frame
- Some things most easily calculated or described in certain reference frames
  - Newton's law
  - Aircraft attitude
  - Aerodynamic forces/torques
  - Accelerometers, rate gyros
  - GPS
  - Mission requirements

Rotation of Reference Frame



$$\mathbf{p} = p_x^0\mathbf{i}^0 + p_y^0\mathbf{j}^0 + p_z^0\mathbf{k}^0$$

$$\mathbf{p} = p_x^1\mathbf{i}^1 + p_y^1\mathbf{j}^1 + p_z^1\mathbf{k}^1$$

$$p_x^1\mathbf{i}^1 + p_y^1\mathbf{j}^1 + p_z^1\mathbf{k}^1 = p_x^0\mathbf{i}^0 + p_y^0\mathbf{j}^0 + p_z^0\mathbf{k}^0$$

$$\mathbf{p}^1 \triangleq \begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix} = \begin{pmatrix} \mathbf{i}^1\cdot\mathbf{i}^0 & \mathbf{i}^1\cdot\mathbf{j}^0 & \mathbf{i}^1\cdot\mathbf{k}^0 \\ \mathbf{j}^1\cdot\mathbf{i}^0 & \mathbf{j}^1\cdot\mathbf{j}^0 & \mathbf{j}^1\cdot\mathbf{k}^0 \\ \mathbf{k}^1\cdot\mathbf{i}^0 & \mathbf{k}^1\cdot\mathbf{j}^0 & \mathbf{k}^1\cdot\mathbf{k}^0 \end{pmatrix} \begin{pmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{pmatrix}$$

$$\mathbf{p}^1 = \mathcal{R}_0^1\mathbf{p}^0 \quad \text{where} \quad \mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(rotation about $\mathbf{k}$ axis)

Rotation of Reference Frame

Right-handed rotation about $\mathbf{j}$ axis:

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Right-handed rotation about $\mathbf{i}$ axis:

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Orthonormal matrix properties:

P.1. $(\mathcal{R}_a^b)^{-1} = (\mathcal{R}_a^b)^\top = \mathcal{R}_b^a$

P.2. $\mathcal{R}_b^c \mathcal{R}_a^b = \mathcal{R}_a^c$

P.3. $\det\bigl(\mathcal{R}_a^b\bigr) = 1$

Particle/Rigid Body Rotation

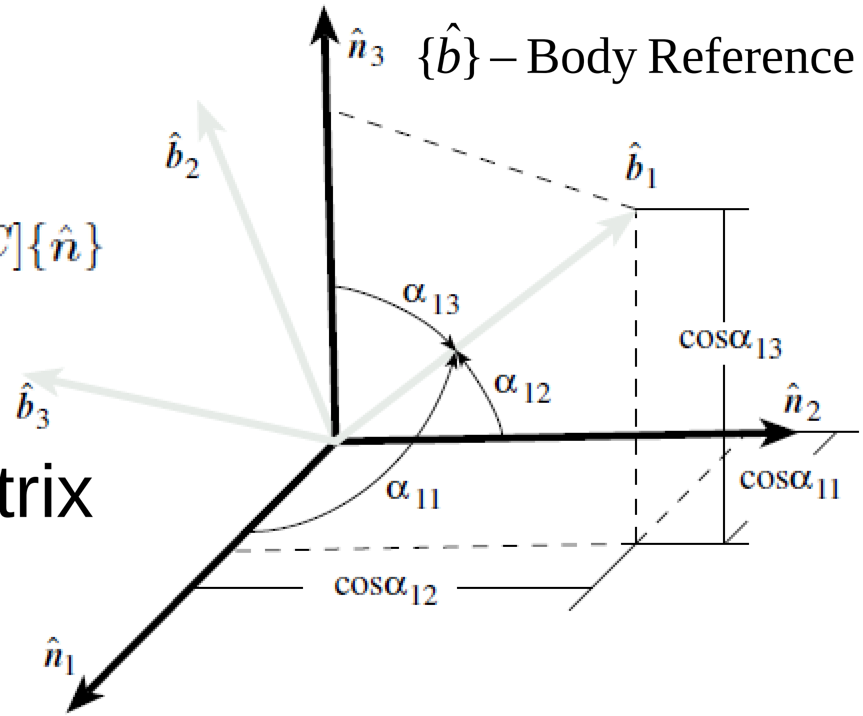
- One can say then that a Rigid Body is essentially a Reference Frame (RF). The translation of the origin of the RF describes the translational position. The specific orientation of the axes wrt to a chosen Inertial Reference provides the angular position.

$$\{\hat{n}\} \equiv \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_2 \end{Bmatrix} \qquad \{\hat{b}\} \equiv \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_2 \end{Bmatrix}$$

$\{\hat{n}\}$ – Inertial Reference

$\{\hat{b}\}$ – Body Reference

$$\{\hat{b}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix} \{\hat{n}\} = [C]\{\hat{n}\}$$



[*C*] – Direction Cosine Matrix

$$C_{ij} = \cos(\angle\, \hat{b}_i, \hat{n}_j) = \hat{b}_i \cdot \hat{n}_j$$

Euler Angles

- Need way to describe attitude of aircraft
- Common approach: Euler angles

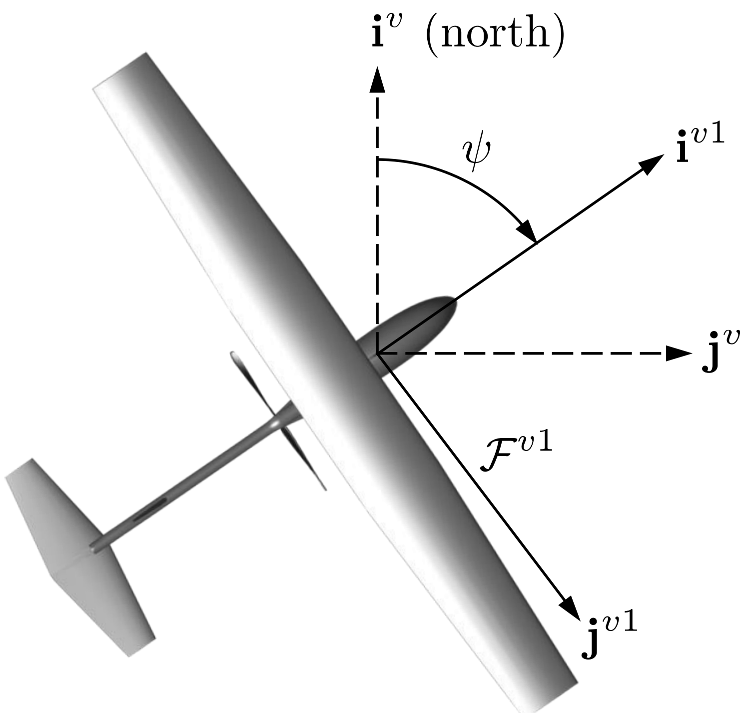
$$\psi\text{: heading (yaw)}$$

$$\theta\text{: elevation (pitch)}$$

$$\phi\text{: bank (roll)}$$

- Pro: Intuitive
- Con: Mathematical singularity
  - Quaternions are alternative for overcoming singularity

Vehicle-1 Frame

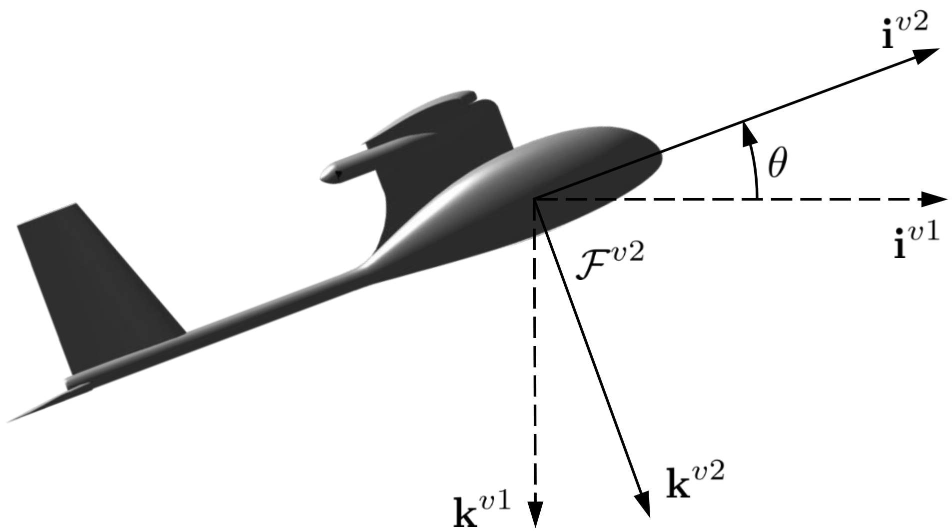


$$\mathcal{R}_v^{v1}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{p}^{v1} = \mathcal{R}_v^{v1}(\psi)\mathbf{p}^v$$

ψ: heading

Vehicle-2 Frame

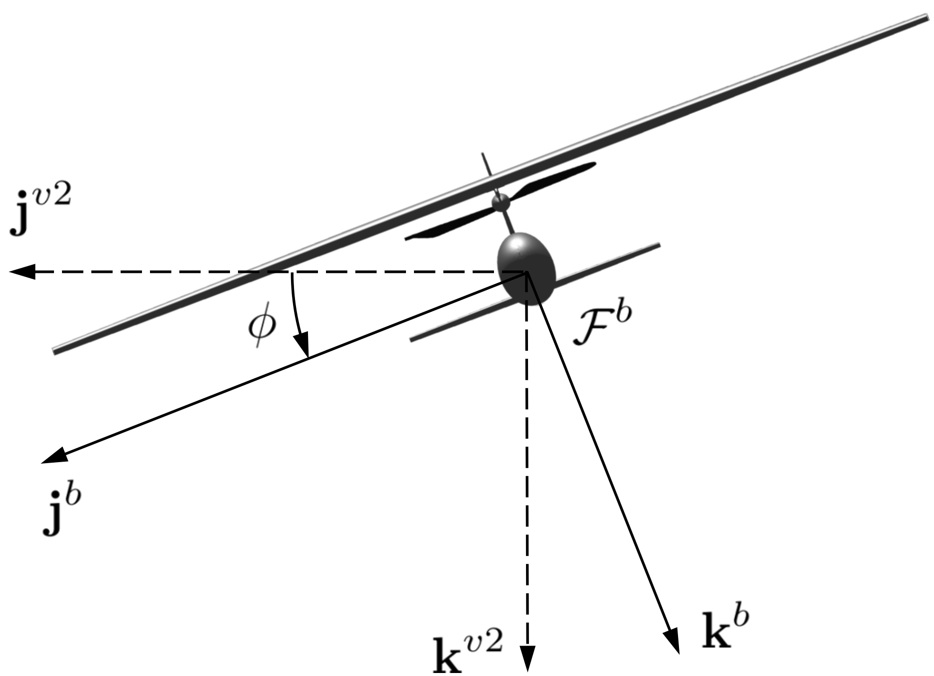


$$\mathcal{R}_{v1}^{v2}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathbf{p}^{v2} = \mathcal{R}_{v1}^{v2}(\theta)\mathbf{p}^{v1}$$

θ: elevation (pitch)

Body Frame



$$\mathcal{R}_{v2}^b(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

$$\mathbf{p}^b = \mathcal{R}_{v2}^b(\phi)\mathbf{p}^{v2}$$

ϕ: bank (roll)

Inertial Frame to Body Frame Transformation

$$
\begin{aligned}
\mathcal{R}_v^b(\phi,\theta,\psi) &= \mathcal{R}_{v2}^b(\phi)\mathcal{R}_{v1}^{v2}(\theta)\mathcal{R}_v^{v1}(\psi) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{c}_\theta \mathrm{s}_\psi & -\mathrm{s}_\theta \\ \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{c}_\theta \\ \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{c}_\theta \end{pmatrix}
\end{aligned}
$$

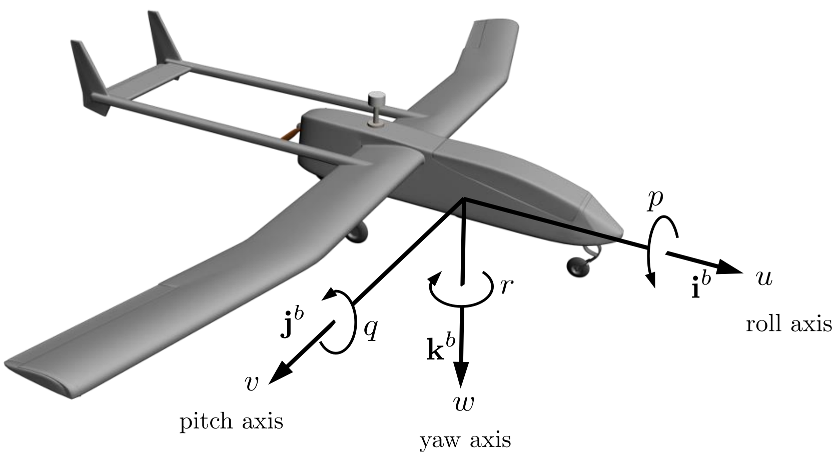
$$
\mathbf{p}^b = \mathcal{R}_v^b(\theta)\mathbf{p}^v
$$

Rotational Kinematics

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \mathcal{R}^b_{v2}(\phi) \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathcal{R}^b_{v2}(\phi)\mathcal{R}^{v2}_{v1}(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

$$= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$



Inverting gives

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Differentiation of a Vector

$$\mathbf{p} = p_x \mathbf{i}^b + p_y \mathbf{j}^b + p_z \mathbf{k}^b$$

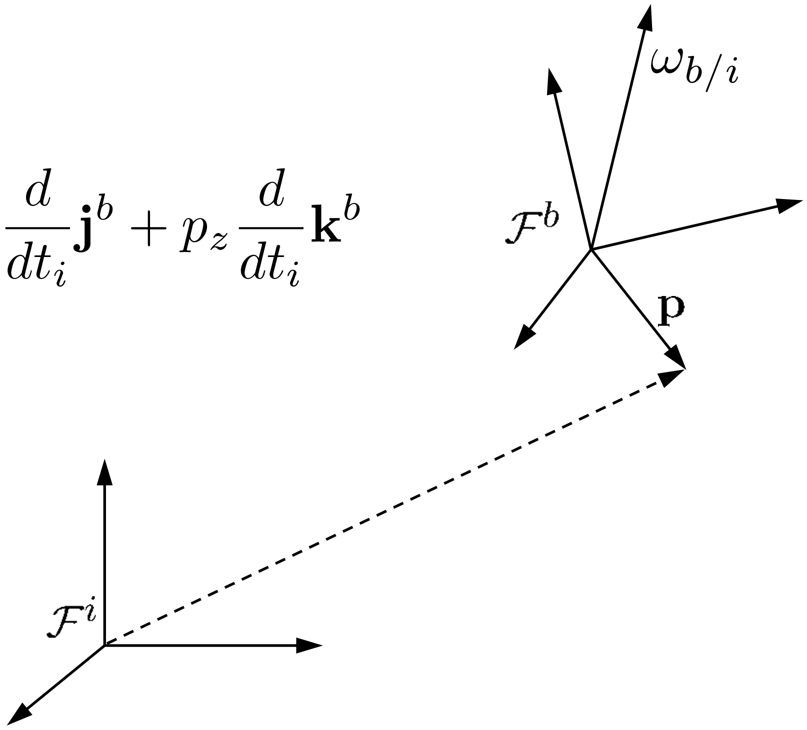
$$\frac{d}{dt_i}\mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b + p_x \frac{d}{dt_i}\mathbf{i}^b + p_y \frac{d}{dt_i}\mathbf{j}^b + p_z \frac{d}{dt_i}\mathbf{k}^b$$

$$\frac{d}{dt_b}\mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b$$

$$\dot{\mathbf{i}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{i}^b$$

$$\dot{\mathbf{j}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{j}^b$$

$$\dot{\mathbf{k}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{k}^b$$



$$\begin{aligned} p_x \dot{\mathbf{i}}^b + p_y \dot{\mathbf{j}}^b + p_z \dot{\mathbf{k}}^b &= p_x(\boldsymbol{\omega}_{b/i} \times \mathbf{i}^b) + p_y(\boldsymbol{\omega}_{b/i} \times \mathbf{j}^b) + p_z(\boldsymbol{\omega}_{b/i} \times \mathbf{k}^b) \\ &= \boldsymbol{\omega}_{b/i} \times \mathbf{p} \end{aligned}$$

$$\frac{d}{dt_i}\mathbf{p} = \frac{d}{dt_b}\mathbf{p} + \boldsymbol{\omega}_{b/i} \times \mathbf{p}$$

Let $\{\boldsymbol{b}\}$ have an angular velocity $\boldsymbol{\omega}$ and be expressed as:

$$\omega = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3$$

Then $$\frac{{}^{\mathcal{N}}d}{dt}\{\hat{b}_i\} = \cancel{\frac{{}^{\mathcal{B}}d}{dt}\{\hat{b}_i\}} + \omega \times \{\hat{b}_i\}$$

skew-symmetric cross product operator

Thus $$\boxed{\frac{{}^{\mathcal{N}}d}{dt}\{\hat{b}\} = -[\tilde{\omega}]\{\hat{b}\}}$$

$$[\tilde{x}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

But LHS $$\frac{{}^{\mathcal{N}}d}{dt}([C]\{\hat{n}\}) = \frac{d}{dt}([C])\{\hat{n}\} + [C]\cancel{\frac{{}^{\mathcal{N}}d}{dt}(\{\hat{n}\})} = [\dot{C}]\{\hat{n}\}$$

Finally $$\left([\dot{C}] + [\tilde{\omega}][C]\right)\{\hat{n}\} = 0$$

$$\boxed{[\dot{C}] = -[\tilde{\omega}][C]}$$

Nine parameter attitude representation

Poisson Kinematic Equation

For the Euler 3-2-1 Sequence

$$[C] = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix}$$

$$\psi = \theta_1 = \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right)$$

$$\theta = \theta_2 = -\sin^{-1}(C_{13})$$

$$\phi = \theta_3 = \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right)$$

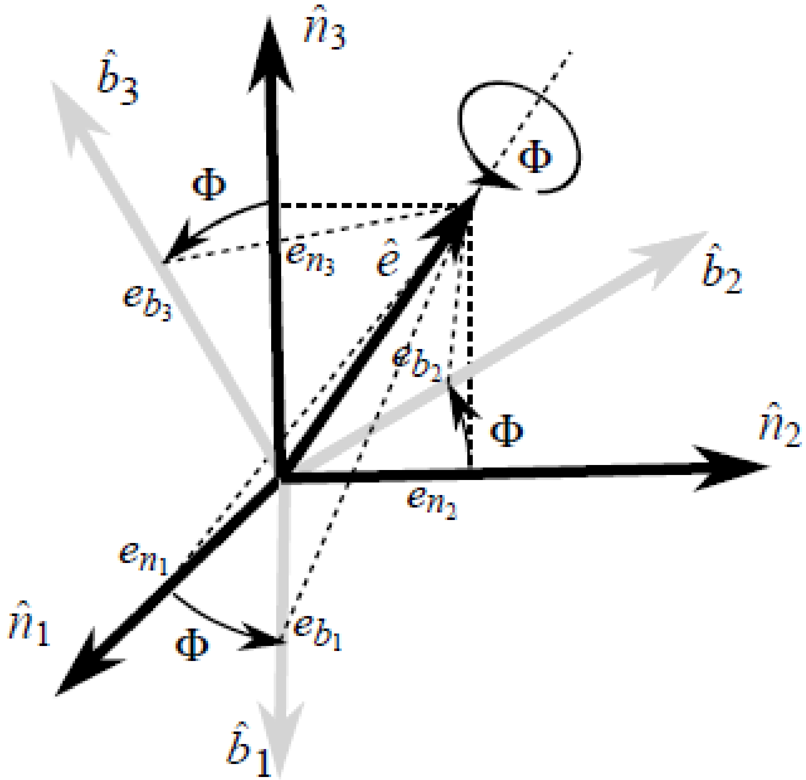
$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Attitude Kinematics Differential Equation

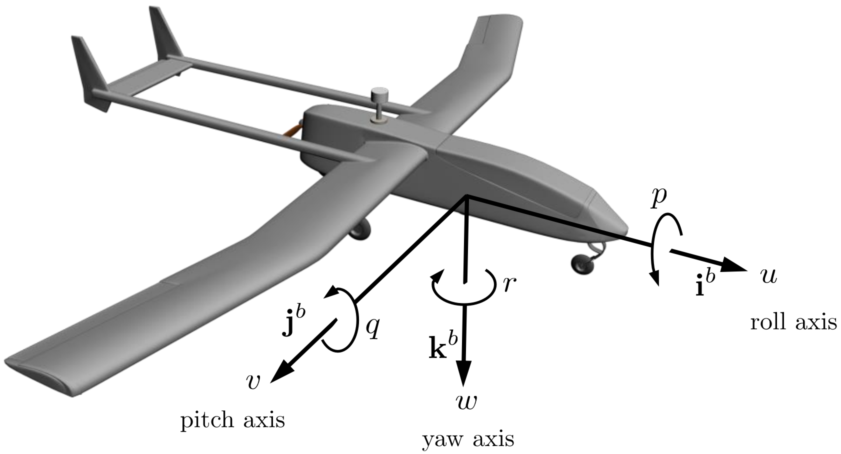
$$\boxed{\begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} 0 & \sin\phi & \cos\phi \\ 0 & \cos\phi\cos\theta & -\sin\phi\cos\theta \\ \cos\theta & \sin\phi\sin\theta & \cos\phi\sin\theta \end{bmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = [B(\psi, \theta, \phi)]\boldsymbol{\omega}}$$

Euler’s Principal Rotation Theorem

Informal Statement: There exists a principal axis about which a single axis rotation through Φ will orient the Inertial axes with the Body axes.



Rotational Dynamics



Newton's 2nd Law:

$$\frac{d\mathbf{h}}{dt_i} = \mathbf{m}$$

- $\mathbf{h}$ is the angular momentum vector
- $\mathbf{m}$ is the sum of all external moments
- Time derivative taken wrt inertial frame

$$\frac{d\mathbf{h}}{dt_i} = \frac{d\mathbf{h}}{dt_b} + \boldsymbol{\omega}_{b/i} \times \mathbf{h} = \mathbf{m}$$

Expressed in the body frame,

$$\frac{d\mathbf{h}^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \mathbf{h}^b = \mathbf{m}^b$$

Rotational Dynamics

$$\frac{d\mathbf{h}^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \mathbf{h}^b = \mathbf{m}^b$$

Because $\mathbf{J}$ is unchanging in the body frame, $\dfrac{d\mathbf{J}}{dt_b} = 0$ and

$$\mathbf{J}\frac{d\boldsymbol{\omega}_{b/i}^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \left(\mathbf{J}\boldsymbol{\omega}_{b/i}^b\right) = \mathbf{m}^b$$

Rearranging we get

$$\dot{\boldsymbol{\omega}}_{b/i}^b = \mathbf{J}^{-1}\left[-\boldsymbol{\omega}_{b/i}^b \times \left(\mathbf{J}\boldsymbol{\omega}_{b/i}^b\right) + \mathbf{m}^b\right]$$

where $\dot{\boldsymbol{\omega}}_{b/i}^b = \dfrac{d\boldsymbol{\omega}_{b/i}^b}{dt_b} = \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix}$

Inertia matrix

$$\mathbf{J} = \begin{pmatrix} \int (y^2 + z^2)\, d\mathsf{m} & -\int xy\, d\mathsf{m} & -\int xz\, d\mathsf{m} \\ -\int xy\, d\mathsf{m} & \int (x^2 + z^2)\, d\mathsf{m} & -\int yz\, d\mathsf{m} \\ -\int xz\, d\mathsf{m} & -\int yz\, d\mathsf{m} & \int (x^2 + y^2)\, d\mathsf{m} \end{pmatrix}$$

Rotational Dynamics

If the aircraft is symmetric about the $\mathbf{i}^b$-$\mathbf{k}^b$ plane, then $J_{xy} = J_{yz} = 0$ and

$$\mathbf{J} = \begin{pmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{pmatrix}$$

This symmetry assumption helps simplify the analysis. The inverse of $\mathbf{J}$ becomes

$$\mathbf{J}^{-1} = \frac{\mathrm{adj}(\mathbf{J})}{\det(\mathbf{J})} = \frac{\begin{pmatrix} J_y J_z & 0 & J_y J_{xz} \\ 0 & J_x J_z - J_{xz}^2 & 0 \\ J_{xz} J_y & 0 & J_x J_y \end{pmatrix}}{J_x J_y J_z - J_{xz}^2 J_y}$$

$$= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix} \qquad \Gamma \triangleq J_x J_z - J_{xz}^2$$

Rotational Dynamics

$$\dot{\boldsymbol{\omega}}^b_{b/i} = \mathbf{J}^{-1}\left[-\boldsymbol{\omega}^b_{b/i} \times \left(\mathbf{J}\boldsymbol{\omega}^b_{b/i}\right) + \mathbf{m}^b\right]$$

Define $\mathbf{m}^b \triangleq \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} &= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix} \left[\begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix} \right] \\
&= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix} \left[\begin{pmatrix} J_{xz}pq + (J_y - J_z)qr \\ J_{xz}(r^2 - p^2) + (J_z - J_x)pr \\ (J_x - J_y)pq - J_{xz}qr \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix} \right] \\
&= \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr + \Gamma_3 l + \Gamma_4 n \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) + \frac{1}{J_y} m \\ \Gamma_7 pq - \Gamma_1 qr + \Gamma_4 l + \Gamma_8 n \end{pmatrix}
\end{aligned}$$

Γ's are functions of moments and products of inertia

Equations of Motion

$$\begin{pmatrix}\dot{p}_n\\ \dot{p}_e\\ \dot{p}_d\end{pmatrix} = \begin{pmatrix}\mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi\\ \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi\\ -\mathrm{s}_\theta & \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{c}_\theta\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}$$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix} = \begin{pmatrix}rv - qw\\ pw - ru\\ qu - pv\end{pmatrix} + \frac{1}{\mathsf{m}}\begin{pmatrix}f_x\\ f_y\\ f_z\end{pmatrix},$$

$$\begin{pmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{pmatrix} = \begin{pmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta}\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}$$

$$\begin{pmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{pmatrix} = \begin{pmatrix}\Gamma_1 pq - \Gamma_2 qr\\ \Gamma_5 pr - \Gamma_6(p^2 - r^2)\\ \Gamma_7 pq - \Gamma_1 qr\end{pmatrix} + \begin{pmatrix}\Gamma_3 l + \Gamma_4 n\\ \frac{1}{J_y} m\\ \Gamma_4 l + \Gamma_8 n\end{pmatrix}$$

External Forces and Moments

$$\mathbf{f} = \mathbf{f}_g + \mathbf{f}_a + \mathbf{f}_p$$

$$\mathbf{m} = \mathbf{m}_a + \mathbf{m}_p$$

gravitational, aerodynamic, propulsion

Gravity Force

$$\mathbf{f}_g^v = \begin{pmatrix} 0 \\ 0 \\ \mathsf{m}g \end{pmatrix}$$

expressed in vehicle frame

$$\begin{aligned} \mathbf{f}_g^b &= \mathcal{R}_v^b \begin{pmatrix} 0 \\ 0 \\ \mathsf{m}g \end{pmatrix} \\ &= \begin{pmatrix} -\mathsf{m}g \sin\theta \\ \mathsf{m}g \cos\theta \sin\phi \\ \mathsf{m}g \cos\theta \cos\phi \end{pmatrix} \end{aligned}$$

expressed in body frame